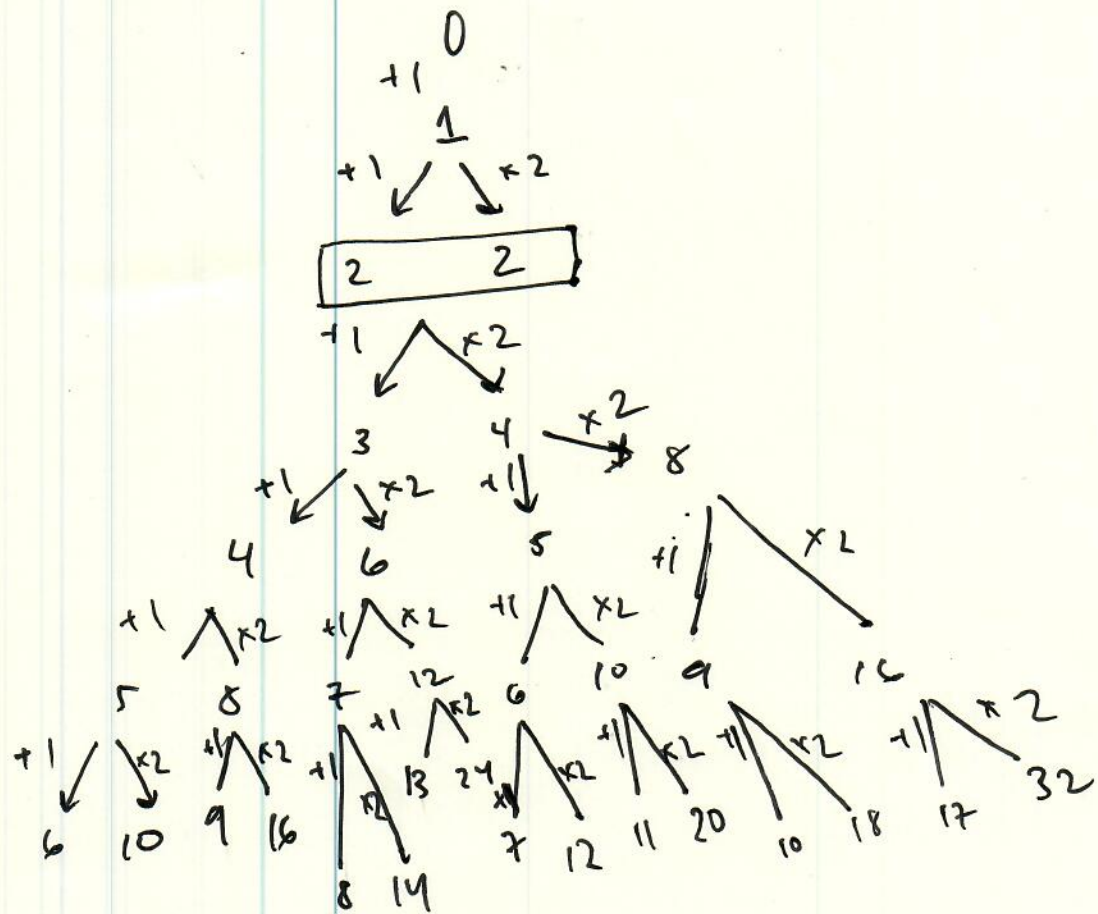
Observations!



Shortest paths to different values of n

$N=1 \longrightarrow 0+1$

$N=2 \longrightarrow (0+1)\times 2$

$N=3 \longrightarrow ((0+1)\times 2)+1$

$N=4 \longrightarrow ((0+1)\times 2)\times 2$

$N=5 \longrightarrow (((0+1)\times 2)\times 2)+1$

$N=6 \longrightarrow (((0+1)\times 2)+1)\times 2$

Looking at the shortest ways to get to different values of n helps us see a pattern.

Proof:

The shortst way to get to an even number is to do x2 the number $\frac{n}{2}$. This may not be true for 2 but it is true for any $n > 2$ where n is even. With $n = 2$, it is equal to the amount of moves of the seqence $(0+1)+1$. The shortest way to get to an odd number is to do +1 of a number $n-1$.

Using these observations we can make the following recursive functions.

when n is odd -

$$f_n = f_{n-1} + 1$$

when n is even -

$$f_n = f_{(\frac{n}{2})} \times 2$$

$$f_1 = 0 + 1$$

example $n = 11$

$$f_{11} = f_{10} + 1 \rightarrow 1+(2\times(1+(2\times(2\times(0+1)))))$$

$$f_{10} = f_5 \times 2 \rightarrow 2\times(1+(2\times(2\times(0+1))))$$

$$f_5 = f_4 + 1 \rightarrow 1+(2\times(2\times(0+1)))$$

$$f_4 = f_2 \times 2 \rightarrow f_4 = 2\times(2\times(0+1))$$

$$f_2 = f_1 \times 2 \longrightarrow f_2 = 2\times(0+1)$$

example

$n = 8$

$$f_8 = f_4 \times 2 \rightarrow (((0+1)\times 2)\times 2)\times 2$$

$$f_4 = f_2 \times 2 \rightarrow ((0+1)\times 2)\times 2$$

$$f_2 = f_1 \times 2 \rightarrow (0+1)\times 2$$

Observation!

We can also make a function for least number of button pressed for any number n.

$N_1 = 1$

$N_2 = 2 \quad N_{\frac{2}{2}}+1$

$N_3 = 3 \quad N_{3-1}+1$

$N_4 = 3 \quad N_{\frac{4}{2}}+1$

$N_5 = 4 \quad N_{5-1}+1$

$N_6 = 4 \quad N_{\frac{6}{2}}+1$

$N_7 = 5 \quad N_{7-1}+1$

$N_8 = 4 \quad N_{\frac{8}{2}}+1$

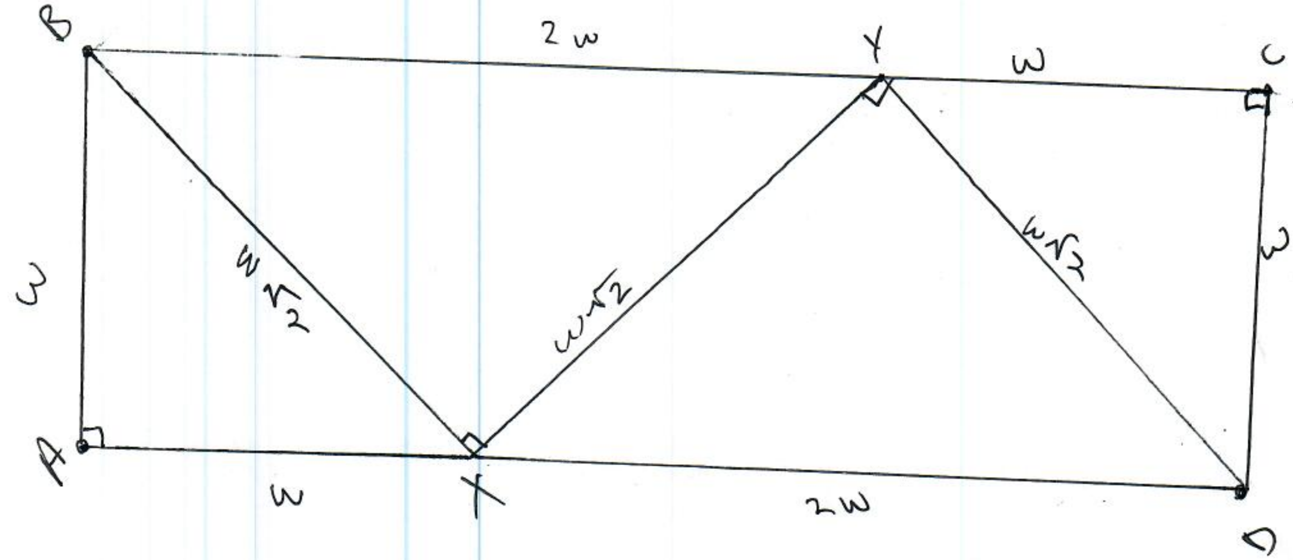
Since each odd number i's +1 the number before it the function for any odd n would be

$$N_n = N_{n-1} + 1$$

Since each even number is x2 the number ~~before~~ $\frac{n}{2}$ the function for any even n would be

$$N_n = N_{\frac{n}{2}} + 1$$

(A)



Since line BX has slope -1, the ratio of height to width of the triangle ΔBAX must be $\frac{1}{1}$

$$\frac{1}{1} = \frac{w}{AX} \rightarrow AX = w$$

This triangle is a right triangle because of perpendicular lines so:

$$BX = \sqrt{BA^2 + AX^2} = w\sqrt{2}$$

Since ~~line~~ XY has slope 1, the ratio of height to width of the triangle ΔBXY must be $\frac{1}{1}$

$$\frac{1}{1} = \frac{BX}{XY} = \frac{w\sqrt{2}}{w\sqrt{2}} \rightarrow XY = w\sqrt{2}$$

Since line YD has slope -1, the ratio of height to width of triangle ΔXYD must be $\frac{1}{1}$

$$\frac{1}{1} = \frac{XY}{YD} = \frac{w\sqrt{2}}{w\sqrt{2}} \rightarrow YD = w\sqrt{2}$$

This is a right triangle because of perpendicular lines so:

$$\sqrt{XY^2 + YD^2} = XD = 2w$$

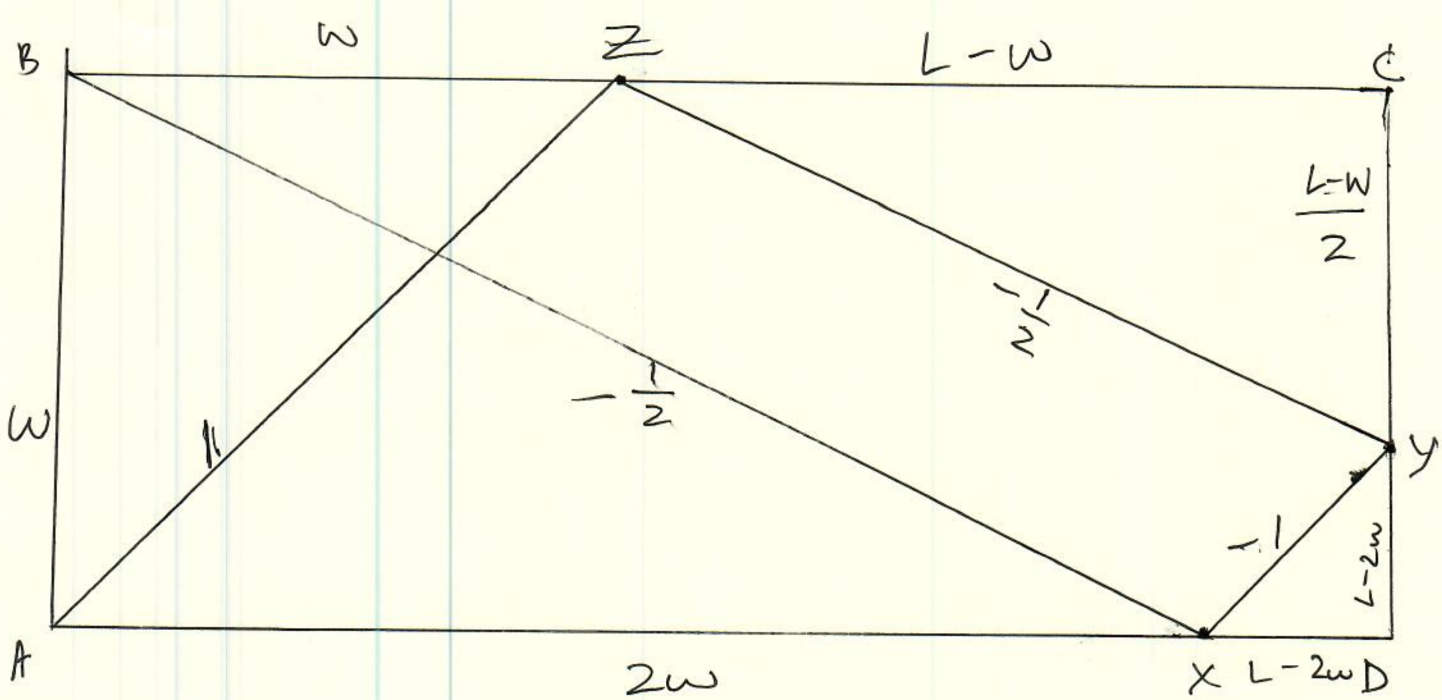
$$AX + XD = AD$$

$$w + 2w = AD$$

$$AD = 3w$$

AD is the length so:

$$\frac{\text{length}}{\text{width}} = \frac{3w}{w} = \boxed{\frac{3}{1}}$$



(B)

Since line BX has slope $-\frac{1}{2}$ the ratio of the height to the width of right triangle $\triangle BAX$ must be $\frac{1}{2}$.

$$\frac{1}{2} = \frac{w}{AX}$$

$$AX = 2w$$

Since AZ has a slope of 1 the ratio of the height and width of the right triangle $\triangle ZBA$ is $\frac{1}{1}$.

$$\frac{1}{1} = \frac{w}{BZ}$$

$$BZ = w$$

If L is the length of the triangle

$$ZC = L - w$$

Since ZY has a slope of $\frac{-1}{2}$ the ratio of height and width of the right triangle $\triangle ZCY$ must be $\frac{1}{2}$.

Mathcamp #2

$$\frac{CY}{L-W} = \frac{1}{2}$$

$$2CY = L-W$$

$$CY = \frac{L-W}{2}$$

$XY = L - 2w$

Since XY has a slope of $\frac{1}{1}$ the ratio of height and width of the right triangle ΔXDY must be $\frac{1}{1}$.

$$\frac{1}{1} = \frac{L-2w}{DY}$$

$$DY = L - 2w$$

Since this is a rectangle

$BA \cong CD$

$$w = \frac{L-w}{2} + L - 2w$$

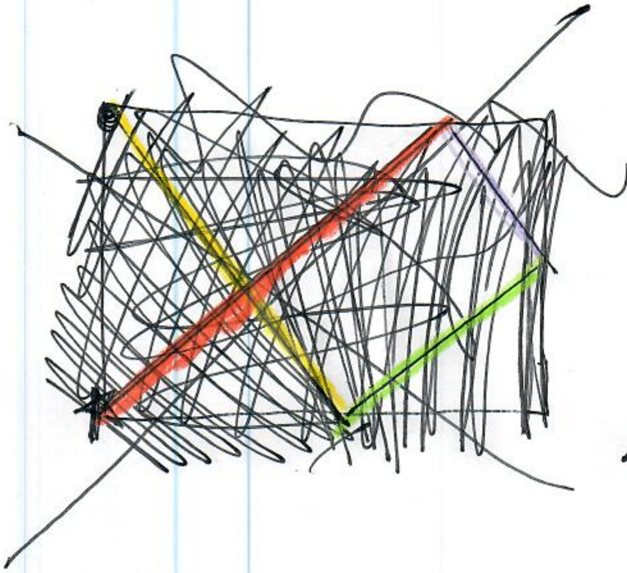
$$7W = 3L$$

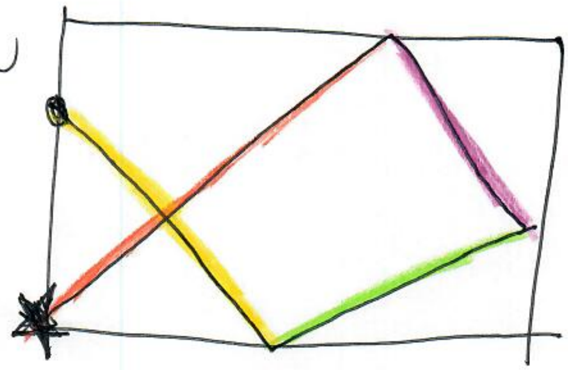
$$\boxed{\frac{L}{W} = \frac{7}{3}}$$

Terms

2c

Cycle: 4 consecutive bounces





final line: last line in a cycle

ex. yellow line in the picture above

final position: last point touched in a cycle

ex. point with the circle in the picture above

start position: ~~final position~~ first point in a cycle

ex. star in the picture above

N: number of cycles

Claim: The trajectory of the ball will approach being a parralelogram between points: $(0,2)$, $(1,3)$, $(5,1)$, $(4,0)$ as the number of bounces $\to \infty$

Proof:

We can think of the bounces in cycles. The start position of the cycle determines the trajectory for the bounces in that cycle and the start of the next cycle.

We can represent the start position with X

X = distance between start position and point B

The series of bounces of each cycle will be repeated again for the next cycle since the ball always ends with the up & right which then bounces at slope 1. The only exeption is the very first bounce which just starts of with bounce slope = 1 without any previous bounce.

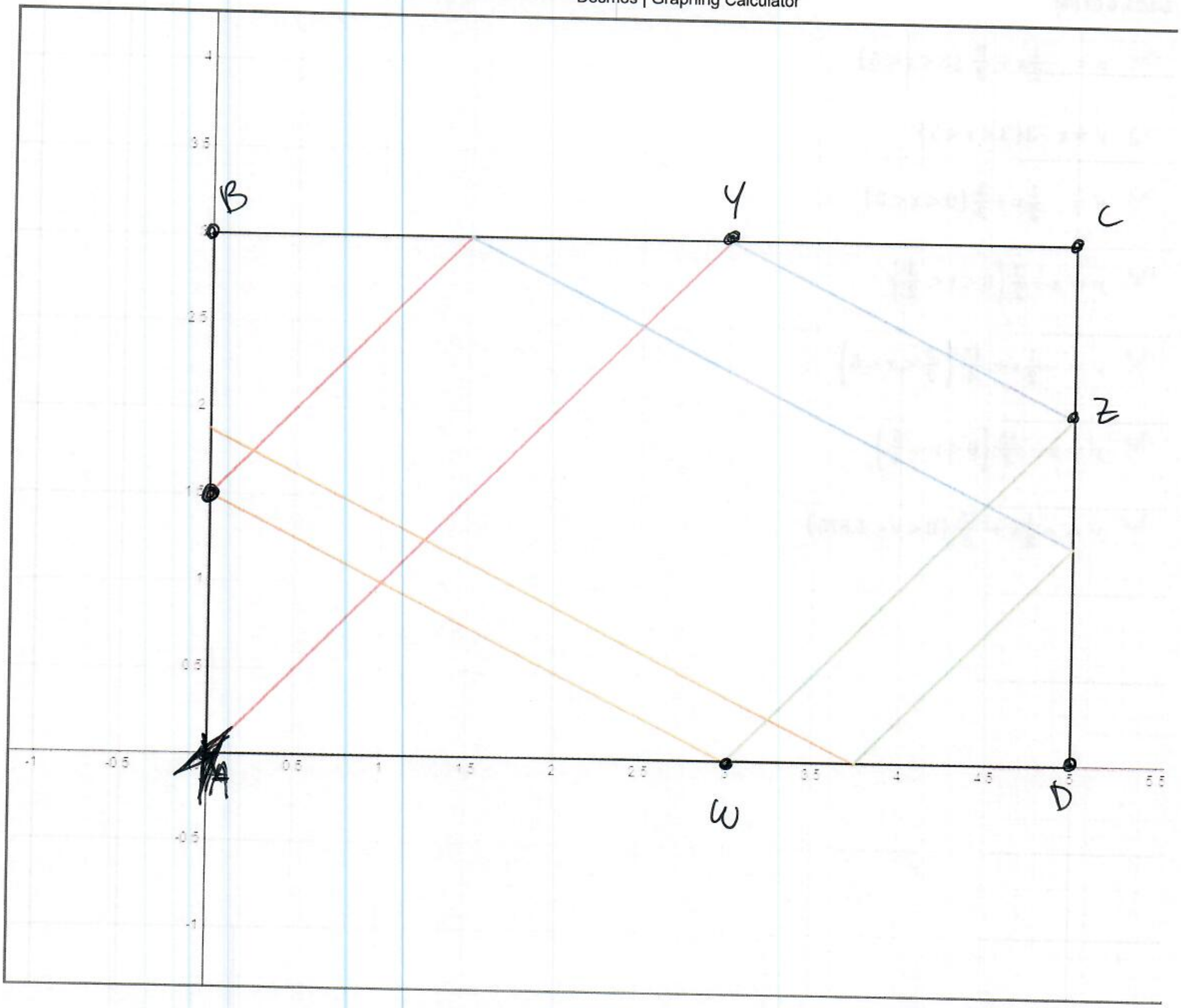
Since we know that the series of bounces doesn't change for every cycle, we can create a function for the distance between point B and the starting point of each cycle.

Then we can prove that this function approaches 1 but ~~approaches~~ never crosses 1. Oncy $y=1$ is proved as ~~the~~ an asymptote of the function, we know that the starting position of the ball will keep on approaching $(0,2)$ since $3-1=2$. The closer the distance is to 1, the smaller the starting point is to 2.

A function can be created for $f(N+1)$ where N equals the n^{th} cycle the starting point is starting. This function will then give the distance between the starting point of the cycle and point B.

We can create this function by doing manipulations needed on the previous x to find the next x.

We can prove it works for the first cycle since all the cycles must follow the



$x = 0 \; \{0 < y < 3\}$

$x = 5 \; \{0 < y < 3\}$

$y = 3 \; \{0 < x < 5\}$

$y = 0 \; \{0 < x < 5\}$

$y = x \; \{0 < y < 3\}$

same pattern.

REFER TO DIAGRAM

$BY = BA$ because XY slope $= 1$

start

$BY = X$

$5 - BY = YC$

$5 - X = YC$

$CZ = \frac{YC}{2}$

$CZ = \frac{5-X}{2}$

$3 - CZ = DZ$

$3 - \left(\frac{5-X}{2}\right) = DZ$

$\frac{1+X}{2} = DZ$

$DZ = DW$ because WZ slope $= 1$

$DW = \frac{1+X}{2}$

$5 - DW = AW$

$5 - \left(\frac{1+X}{2}\right) = AW$

$\frac{9-X}{2} = AW$

$AE = \frac{AW}{2}$ because BW slope $= \frac{-1}{2}$

$AE = \frac{9-X}{4}$

$3 - AE = EB$

$3 - \left(\frac{9-X}{4}\right) = EB$

$\frac{3+X}{4} = EB$

$EB =$ new X

$$\boxed{f(N+1) = \frac{3+f(N)}{4}}$$

The first $f(N)$ is $\boxed{f(1) = 3}$

1 is the asymptote of $f(N+1) = \frac{3+f(N)}{4}$

because f(N) would have to be less than

or equal to 1 for it to cross y=1.

The numerator woud need to be less than

or equal to 4 since f(1) is greater than

1 which creates an f(1+1) more than

1. which creates an f(2+1) greater than 1.

The function will keep getting smaller but

will always be greater than 1.

Since as $N \to \infty$, $f(N) \to 1$ each new cycle becomes closer and closer togeter. As each cycle starts getting closer to starting at $(0,2)$, The path becomes more and more similar to the path of starting at $(0,2)$. This means that the ball will infinitely bouncing while getting closer and closer to the path of the ball having its cycle start at $(0,2)$.

3(a)

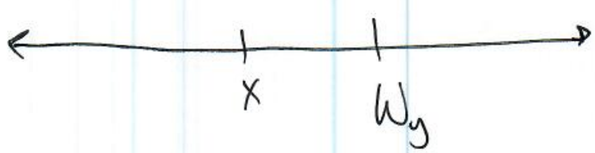
we can treat the scale as a machine that gives out boolean values

Weight is less than x grams — True

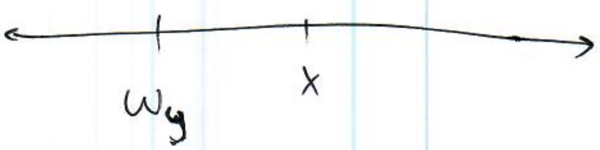
weight is NOT less than x grams — False

For a random integer weight W_y the scale will show either true or false

examples:



display: False



display: True

If W_y is less than x then it will display True

$W_y < x$ — True

What if $W_y + 1 \nless x$? or $W_y + 2 \nless x$

$W_y + 2 \nless x$, $W_y + 1 \nless x$ — False

Every interval we increase W_y by will be an integer so W_y is ~~an~~ still an integer.

So if W_y is True and W_{y+1} is False, this mean $W_{y+1} = X$. ~~st~~ This is because false is everything besides less than, which is more than or equal too.

3(b)

To find such a set in at most 10,000 weighings, we can use a binary search method and the boolean values from part 3(a).

We can show that the value that takes the most amount of weighings with this method will be less than 10,000.

It is know that the worst case for a binary search are the extreme ends of the set being searched. This is because it will take the most "cuts in half" of the acceptable portions of the set to find either 2 or 6069. source: stackoverflow.com/questions/50290996

The scale will always show "false" for any new value you put on the scale when $X=2$ until you put weight 2 on the scale. The scale will always show "true" for any new value you put on the scale when $x=6069$ until you put weight 6069 on the scale.

We can simulate the binary search for $x = 2$ and $x = 6069$ to show that it can be done in less than 10,000 weighings. Since these two numbers ~~we know~~ are the numbers we know will take the most amount of weighings to find, all the other numbers must take less weighings to find. This means that if 2 & 6069 take less than 10,000 weighings, all the other number will take less than 10,000 weighings.

The medians of each set will be floored so that the next start/end of the next set will be the next/previous consecutive integer after the floored median.

Finding X=6069

(2...6069) - median = 3035 - display: True 1
we then know that our X is ~~greater~~ greater than 3035 so
we make the acceptable set start from 3036

(3036... 6069) - median = 4552 - display: True 2

(4553... 6069) - median = 5311 - display: True 3

(5312... 6069) - median = 5690 - display: True 4

(5691... 6069) - median = 5880 - display: True 5

(5881 ... 6069) median = 6021 - display: True 6

(6022 ... 6069) median = 6045 - display: True 7

(6046 ... 6069) median = 6057 - display: True 8

(6058 .. 6069) median = 6063 - display: True 9

(6064 ... 6069) median = 6066 display: True 10

(6067 ... 6069) median = 6068 display: True 11

(~~6068~~ 6069) - ~~median = 6068~~ median = 6069 display = False 12

Finding X = 2

(2...6069) - median = 3035 - display: False

we then know that our X is ~~great~~ less than 3035 so we make the ending of the next acceptable set less than 3035.

(2...3034) - median = 1518 - display: False

(2... 1517) - median = 759 - display: False

(2...758) - median = 380 - display: False

(2...379) - median = 190 - display: False

(2...189) - median = 95 - display: False

(2...94) - median = 48 - display: False

(2... 47) - median = 24 - display: False

(2... 23) - median = 12 - display: False

(2... 11) - median = 6 - display: False

(2...5) - median = 3 - display: False

(2) - median = 2 - display: False

2 is the smallest value we can have so we know that it is eqal to X because False means greater to or equal to.

To make the different specific weights you need to put on the scale, we can create an algorithm.

We know there are equal number of 1 pound weights and 2 pound weights and by weight 6069 we must use all the weights.

As we create the combinations of 1s and 2s to create our wanted number, we must try to get the number of 1s & 2s as equal as possible.

1	2	number
1	0	1
0	1	2
1	1	3
2	1	4
1	2	5
2	2	6
		⋮
2022	2023	6068
2023	2023	6069

We can see a pattern where whenever the number is a multiple of ~~three~~ three, the number of 2s & 1s are equal. This is because 2 + 1 = 3 so if there is

$n(2+1)$ it will equal $3n$.

The number after a multiple of 3 will be +1 then the multiple of ~~three~~ 3 and the number 2 after will be +2 the multiple of 3. That is how we can form the numbers in between.

We now have what we need to make an algorithm to form any weight N out of only 1 & 2 pound weights:

Find the nearest multiple of 3 that is less than or equal to N

If N ~~equals~~ is a multiple of 3 then $\frac{N}{3}$ is the number of 2s and number of 1s you need to form N.

If N is not a multible of 3

Then ~~do~~ divide the ~~non-mult~~ nearest multiple of 3 less than N by 3. We can call this multible of 3 : T.

Divide $\frac{T}{3}$.

If N is 1 more than a multiple of 3 then there are $\frac{T}{3}+1$ 1s and $\frac{T}{3}$ 2s.

If N is 2 more than a multiple of 3 then there are $\frac{T}{3}$ 1s and $\frac{T}{3}+1$ 2s.

4A) There is a $\boxed{1/3}$ probability the red key will be in the first treasure

Proof: The only "valid" assignment that we have to start, are the 3 chests in the starting room. This is because we have no keys so we can't go anywhere else. If a red key and blue key are placed in any of three chests with one key taking up one chest, then 1 chest will have red, 1 will have blue, and 2 will be empty. Thus the probability of chosing the red key is $1/3$, chosing the ~~1/3 & 0~~ 1 out of 3 chests with a red key.

4b) $\boxed{\frac{5}{21}}$

Case 1: red in first chest in starting room then blue in same room or red room

Case 2: ~~red~~ blue in first chest in starting room then red in the blue room

$$\frac{\text{case }1}{\text{case }1 + \text{case }2} \cdot \frac{1}{3} = \frac{5}{21}$$

4c)

Insight: ∞ rooms with 1 chest each

5(a): Claim: Travis will always win when N is odd. Narmada will always win when N is even.

Proof: For ~~each~~ number lines with N spots, there are the ordered pairs of ~~&~~ the possible positions of Narmada and Travis. The ordered pairs are still valid if there is (a, b) & then (b, a). This means the orderd pairs are all the different permutations of positions on the number line. The number of these ordered pairs is simply given by:

$$N^2$$

since (a, b) is different than (b, a). Each number ~~0~~1 through N on both number lines are used.

The person that wins is the person that uses up the last ordered pair available.

1 ordered pair is already used up at the beginning because of the starting position $(1, N)$.

Now there are $N^2 - 1$ available pairs.

Narmada will go on every odd turn (1st, 3rd, 5th ....) since it starts on the first turn and then alternates.

Travis will go on all the even turns.

This means if N^2-1 is odd Narmada ~~will all~~ ~~and if it is even Th~~ be the last turn since she goes on odd turns and would win. Vice versa for Travis if N^2-1 is even.

N^2-1 is odd means N^2 will be even

N^2-1 is even means N^2 will be odd.

If N^2 is even, N is even

If N^2 is odd, N is odd

5(b) Claim: When N is odd, if $N-1$ has only 1 factor of 2, then Travis wins. If $n-1$ has more than 1 factor of 2, Narmada wins.

When N is even, if N has 1 factor of 2, Travis wins. If N has more than 1 factor of 2, Narmada wins.

Proof:

Chosing 2 distinct positions for Narmada and Travis to stand is just $\binom{N}{2}$

$$\binom{N}{2} = \text{Total number of pairs}$$

$$\binom{N}{2} = \frac{(n!)}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

1 pair is used in starting position so from turn 1 there are $\frac{n(n-1)}{2} - 1$ possible pairs.

Narmada goes on all the odd turns so when $\frac{n(n-1)}{2} - 1$ is odd, Narmada wins.

Travis goes on all the even turns so when $\frac{n(n-2)}{2} - 1$ is even, Travis wins.

$\frac{n(n-1)}{2}$ will be even if either n or ~~n-~~1 has more than one factor of 2.

$\frac{n(n-1)}{2}$ will be odd if there is only one factor of 2 in ~~n~~ or n-1.

This means if N is odd n-1 must determine the number of factors of 2. If there is more than 1 factor of 2 it will be even and then $\frac{N(N-1)}{2} - 1$ will be odd so Narmada wins. If N is odd but N-1 is even with one factor it will be odd and $\frac{n(n-1)}{2}$ will be even and Travis will win.

If N is even n must determine the number of factors of 2. If there is ~~one~~ more than 1 factor of 2 it will be even and $\frac{n(n-1)}{2} - 1$ will be odd so Narmada will win. If there is only 1 factor of 2 in N the $\frac{N(n-1)}{2}$ will be even and Travis will win.

5(c) Claim: Narmada will always win for any N.

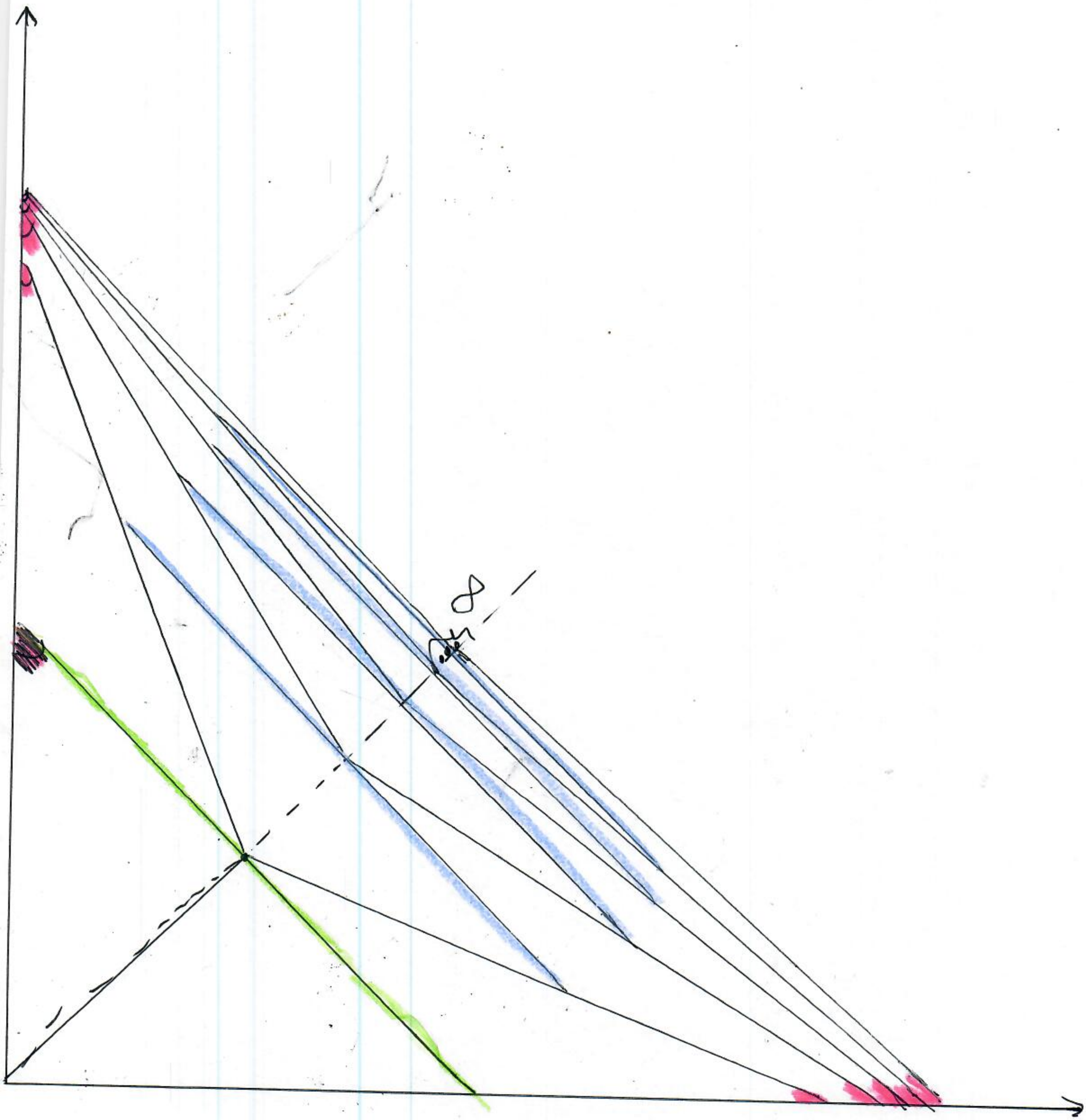
Proof: The number of ordered pairs are given by $n \cdot (n-1)$ because it is the same as part 5(b) except you can have (a, b) & (b, a) so there is no need to divide duplicate combinations.

$n(n-1)$ will always be even because if n is even $n-1$ will be odd. If n is odd $n-1$ will be even. Odd x Even = Even

Then to account for the starting position we have to minus 1

$n(n-1) - 1$ odd

Narmada will always win if it ends in an odd numbered turn.



In order to get to (2023, 2023) we need to traverse the first quadrent in the up and right direction. We would need to travel with slope = 1 direction since (2023, 2023) is on line y = x. The only way to do this is to have a length 1 line segment of slope -1 in betweeen the x and y axis. This line is the green line in the diagram. In order for us to traverse higher we must make 2 lines that the next length 1 slope 1 -1 segment can attatch to. This is because the maximum length is 1 so it can't reach all the way to the x and y axis. We need to have safe points that are closer together. We can attatch these lines from the midpoint of slope -1 line and then either to the x or y axis. After drawing the second length 1 and slope -1 line ~~...~~ and attatching it to our safe lines, it is a farther distance away from our axis. This causes the angle between the safe lines and the axis they are attatched to to get bigger since it is a farther distance away from the ~~...~~ axis.

This angle will approach 45°. This is because the green and blue segments ~~cannot be anyt~~ are slope -1 and are getting closer and closer to the safe lines. This means the safe lines will be getting closer and closer to the slope -1 lines. ~~Since~~ This will cause the middle lines to not travel anymore which means they can never reach $(2023, 2023)$.

(a) no boards will ever be enough

(b) it is not possible

(c) no boards will ever be enough.